A FORMULATION OF THE SIMPLE THEORY OF TYPES

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History

From "A Challenge for Mechanized Deduction" [Benzmüller and Kerber 2001]

- Higher-order logic: a far more expressive language than first-order logic.
- Russell's paradox [Russell 1902, 1903]: no self-reference.

$$\text{When } R = \{X \mid X \text{ in } X\}, \ R \text{ in } R?$$

- A type can be a solution [Russell 1908] by differentiating between objects and sets (functions).
- Formalization with a typed λ-calculus [Church 1940]
  - A basis formalizm in modern higher-order theorem provers: TPS, HOL, PVS, LEO, ΩMEGA, λ CL^{A}M
  - Opens the propositions-as-types idea
  - Opens constructive type theory

Overview

- Logical formulas are represented by λ-terms.
- Rules of inference: λ-conversion + α.
- Axioms: propositional calculus, logical functional calculus, number theory
- Proof & Theorems
- Examples
  - Peano's arithmetic
  - Primitive recursion

WELL-FORMED FORMULAS

- *Type symbols*
  - o (propositions) and ι (individuals)
  - $\alpha \to \beta$ for all α, β (functions)
    * α, β, γ: variable or undetermined type symbols
      α':$(\alpha \to \alpha) \to (\alpha \to \alpha)$
- A *formula* is a finite sequence of:
  - improper symbols: λ ( ) .
  - constants: $\mathbf{N}_{o\to o}$ $\mathbf{A}_{o\to o\to o}$ $\Pi_{(\alpha\to o)\to o}$ $\iota_{(\alpha\to o)\to\alpha}$
  - variables: x_α y_α $\cdots$
- *Well-formed* formulas:
  - a single proper symbol (type: its subscript)
  - $\lambda x_\beta.M_\alpha$ for all well-formed M_α (type: $\beta \to \alpha$)
  - $F_{\beta\to\alpha}A_\beta$ for all well-formed $F_{\beta\to\alpha}$, A_β (type: α)
    * capital letters: variable or undetermined well-formed formulas

ABBREVIATIONS (I)

$$\neg A \longrightarrow \mathbf{N}_{o \to o} A_o$$
$$A_o \vee B_o \longrightarrow \mathbf{A}_{o \to o \to o} A_o B_o$$
$$A_o \wedge B_o \longrightarrow \neg(\neg A_o \vee \neg B_o)$$
$$A_o \supset B_o \longrightarrow \neg A_o \vee B_o$$
$$A_o \equiv B_o \longrightarrow (A_o \supset B_o) \wedge (B_o \supset A_o)$$

$$\forall x_\alpha . A_o \longrightarrow \Pi_{(\alpha \to o) \to o} \lambda x_\alpha . A_o$$
$$\exists x_\alpha . A_o \longrightarrow \neg(\forall x_\alpha . \neg A_o)$$
$$\varepsilon x_\alpha . A_o \longrightarrow \iota_{(\alpha \to o) \to \alpha} \lambda x_\alpha . A_o$$

$$\mathbf{Q}_{\alpha \to \alpha \to o} \longrightarrow \lambda x_\alpha . \lambda y_\alpha . \forall f_{\alpha \to o} . f_{\alpha \to o} x_\alpha \supset f_{\alpha \to o} y_\alpha$$
$$A_\alpha = B_\alpha \longrightarrow \mathbf{Q}_{\alpha \to \alpha \to o} A_\alpha B_\alpha$$
$$A_\alpha \neq B_\alpha \longrightarrow \neg(A_\alpha = B_\alpha)$$

ABBREVIATIONS (II)

$$\mathbf{I}_{\alpha\to\alpha} \longrightarrow \lambda x_\alpha . x_\alpha$$

$$\mathbf{K}_{\alpha\to\beta\to\alpha} \longrightarrow \lambda x_\alpha . \lambda y_\beta . x_\alpha$$

$$\mathbf{0}_{\alpha'} \longrightarrow \lambda f_{\alpha\to\alpha} . \lambda x_\alpha . x_\alpha$$

$$i_{\alpha'} \longrightarrow \lambda f_{\alpha\to\alpha} . \lambda x_\alpha . f^i_{\alpha\to\alpha} x_\alpha$$

$$\mathbf{S}_{\alpha'\to\alpha'} \longrightarrow \lambda n_{\alpha'} . \lambda f_{\alpha\to\alpha} . \lambda x_\alpha . f_{\alpha\to\alpha}(n_{\alpha'} f_{\alpha\to\alpha} x_\alpha)$$

$$\mathbf{N}_{\alpha'\to o} \longrightarrow \lambda n_{\alpha'} . \forall f_{\alpha\to o} . \; f_{\alpha\to o}\mathbf{0}_{\alpha'} \supset (\forall x_{\alpha'} . f_{\alpha\to o} x_{\alpha'} \supset f_{\alpha\to o}(\mathbf{S}_{\alpha'\to\alpha'} x_{\alpha'})) \supset f_{\alpha\to o} n_{\alpha'}$$

$$\mathbf{P}_{\alpha'\to\alpha'} \longrightarrow \lambda x_{\alpha'} . \mathbf{P}_{\alpha'\to\alpha'''}(\mathbf{T}_{\alpha''\to\alpha'''}(\mathbf{T}_{\alpha'\to\alpha''} x_{\alpha'}))$$

$$\mathbf{T}_{\alpha'\to\alpha''} \longrightarrow \lambda x_{\alpha'} . \varepsilon x_{\alpha''} . \mathbf{N}_{\alpha''\to o} x_{\alpha''} \wedge x_{\alpha''} \mathbf{S}_{\alpha'\to\alpha'} \mathbf{0}_{\alpha'} = x_{\alpha'}$$

$$\mathbf{P}_{\alpha'''\to\alpha'} \longrightarrow \lambda n_{\alpha'''} . n_{\alpha'''} \left(\lambda p_{\alpha''} . \left\langle \begin{array}{r} \mathbf{S}_{\alpha'\to\alpha'}(p_{\alpha''}(\mathbf{K}_{\alpha'\to\alpha'\to\alpha'} \mathbf{I}_{\alpha'\to\alpha'})\mathbf{0}_{\alpha'}), \\ p_{\alpha''}(\mathbf{K}_{\alpha'\to\alpha'\to\alpha'} \mathbf{I}_{\alpha'\to\alpha'})\mathbf{0}_{\alpha'} \end{array} \right\rangle \right) \langle \mathbf{0}_{\alpha'}, \mathbf{0}_{\alpha'} \rangle \; (\mathbf{K}_{\alpha'\to\alpha'\to\alpha'} \mathbf{0}_{\alpha'}) \; \mathbf{I}_{\alpha'\to\alpha'}$$

$$\langle A_{\alpha'}, B_{\alpha'} \rangle \longrightarrow \omega_{\alpha'\to\alpha'\to\alpha''} A_{\alpha'} B_{\alpha'}$$

$$\omega_{\alpha'\to\alpha'\to\alpha''} \longrightarrow \lambda y_{\alpha'} . \lambda z_{\alpha'} . \lambda f_{\alpha\to\alpha} . \lambda g_{\alpha'} . \lambda h_{\alpha\to\alpha} . \lambda x_\alpha . \; y_{\alpha'}(f_{\alpha\to\alpha} g_{\alpha'} h_{\alpha\to\alpha})(z_{\alpha'}(g_{\alpha'} h_{\alpha\to\alpha}) x_\alpha)$$

Rules of Inference

I*.	$M_\alpha \longrightarrow M_\alpha\{y_\beta/x_\beta\}$	when x_β is not a free variable of M_α and y_β does not occur in M_α. (α-*conversion*)
II*.	$(\lambda x_\beta.M_\alpha)\ N_\beta \longrightarrow M_\alpha\{N_\beta/x_\beta\}$	when the bound variables of M_α are distinct both from x_β and from the free variables of N_β. (β-*conversion*)
III*.	$M_\alpha\{N_\beta/x_\beta\} \longrightarrow (\lambda x_\beta.M_\alpha)\ N_\beta$	when the bound variables of M_α are distinct both from x_β and from the free variables of N_β.
IV.	$F_{\alpha\to o}x_\alpha \longrightarrow F_{\alpha\to o}A_\alpha$	when x_α is not a free variable of $F_{\alpha\to o}$. (*substitution*)
V.	$A_o \supset B_o$ and $A_o \longrightarrow B_o$	(*modus ponent*)
VI.	$F_{\alpha\to o}x_\alpha \longrightarrow \Pi_{(\alpha\to o)\to o}F_{\alpha\to o}$	when x_α is not a free variable of $F_{\alpha\to o}$. (*generalization*)

* any part of a formula

Derived Rules

IV′. $M_o \longrightarrow M_o\{A_\alpha/\text{free } x_\alpha\}$ when the bound variables of M_o other than x_α are distinct from the free variables of A_α.

(by rule I–IV)

VI′. $M_o \longrightarrow \forall x_\alpha . M_o$
(by rule I, III and VI)

- A derivation of VI′

$$M_o \xrightarrow{\text{I}} M'_o \xrightarrow{\text{III}} (\lambda x_\alpha . M'_o) x_\alpha$$
$$\xrightarrow{\text{VI}} \Pi_{(\alpha \to o) \to o}(\lambda x_\alpha . M'_o) \xrightarrow{\text{I}} \Pi_{(\alpha \to o) \to o}(\lambda x_\alpha . M_o)$$

where M'_o is α-equivalent to M_o and x_α is not a bound variable of M_o.

Formal Axioms

1. $p \vee p \supset p$
2. $p \supset p \vee q$
3. $p \vee q \supset q \vee p$
4. $(p \supset q) \supset (r \vee p \supset r \vee q)$

5^α. $\Pi_{(\alpha\to o)\to o} f_{\alpha\to o} \supset f_{\alpha\to o} x_\alpha$

6^α. $(\forall x_\alpha . p \vee f_{\alpha\to o} x_\alpha) \supset p \vee \Pi_{(\alpha\to o)\to o} f_{\alpha\to o}$

7. $\exists x_\iota . \exists y_\iota . x_\iota \neq y_\iota$
8. $\mathbf{N}_{\iota'\to o} x_{\iota'} \supset \mathbf{N}_{\iota'\to o} y_{\iota'} \supset \mathbf{S}_{\iota'\to\iota'} x_{\iota'} = \mathbf{S}_{\iota'\to\iota'} y_{\iota'} \supset x_{\iota'} = y_{\iota'}$

9^α. $f_{\alpha\to o} x_\alpha \supset (\forall y_\alpha . f_{\alpha\to o} y_\alpha \supset x_\alpha = y_\alpha) \supset f_{\alpha\to o}(\iota_{(\alpha\to o)\to\alpha} f_{\alpha\to o})$

$10^{\alpha\beta}$. $(\forall x_\beta . f_{\beta\to\alpha} x_\beta = g_{\beta\to\alpha} x_\beta) \supset f_{\alpha\beta} = g_{\alpha\beta}$

11^α. $f_{\alpha\to o} x_\alpha \supset f_{\alpha\to o}(\iota_{(\alpha\to o)\to\alpha} f_{\alpha\to o})$

* 1–4: propositional calculus, 1–6^α: logical functional calculus, 7–9^α: elementary number theory, $10^{\alpha\beta}$–11^α: classical real number theory

PROOF

- A *proof* of a formula B_o on the assumption of the formulas $A_o^1, A_o^2, \cdots, A_o^n$ is a finite sequence of formulas:
  - one of formulas $A_o^1, A_o^2, \cdots, A_o^n$,
  - a variant[1] of a formal axiom, or
  - obtainable from preceding formulas by a rule of inference,

  which ends with B_o.
- When B_o has a proof on the assumption $A_o^1, A_o^2, \cdots, A_o^n$, we write:

$$A_o^1, A_o^2, \cdots, A_o^n \vdash B_o$$

- Deduction theorem

VII. If $A_o^1, A_o^2, \cdots, A_o^n \vdash B_o$, then $A_o^1, A_o^2, \cdots, A_o^{n-1} \vdash A_o^n \supset B_o$.

[1] A formal theorem obtained by alphabetically changing variables by rules I and IV′.

THEOREMS

12^{α}. $(\forall x_{\alpha}.f_{\alpha\to o}x_{\alpha}) \supset f_{\alpha\to o}y_{\alpha}$

13^{α}. $f_{\alpha\to o}y_{\alpha} \supset \exists x_{\alpha}.f_{\alpha\to o}x_{\alpha}$

14^{α}. $(\forall x_{\alpha}.p \supset f_{\alpha\to o}x_{\alpha}) \supset p \supset \forall x_{\alpha}.f_{\alpha\to o}x_{\alpha}$

15^{α}. $(\forall x_{\alpha}.f_{\alpha\to o}x_{\alpha} \supset p) \supset (\exists x_{\alpha}.f_{\alpha\to o}x_{\alpha}) \supset p$

16^{α}. $x_{\alpha} = x_{\alpha}$

17^{α}. $x_{\alpha} = y_{\alpha} \supset f_{\alpha\to o}x_{\alpha} \supset f_{\alpha\to o}y_{\alpha}$

$18^{\beta\alpha}$. $x_{\alpha} = y_{\alpha} \supset f_{\alpha\to\beta}x_{\alpha} = f_{\alpha\to\beta}y_{\alpha}$

19^{α}. $x_{\alpha} = y_{\alpha} \supset y_{\alpha} = x_{\alpha}$

20^{α}. $x_{\alpha} = y_{\alpha} \supset y_{\alpha} = z_{\alpha} \supset x_{\alpha} = z_{\alpha}$

$21^{\alpha\beta}$. $f_{\beta\to\alpha} = \lambda x_{\beta}.f_{\beta\to\alpha}x_{\beta}$

Peano's Axioms

http://mathworld.wolfram.com/PeanosAxioms.html

- 0 is a number.
- If a is a number, the successor of a is a number.
- 0 is not the successor of a number.
- Two numbers of which the successors are equal are themselves equal.
- (induction axiom) If a set S of numbers contains 0 and also the successor of every number in S, then every number is in S.

Peano's Axioms

- 0 is a number.
  22^{α}. $\mathbf{N}_{\alpha'\to o}\mathbf{0}_{\alpha'}$
- If a is a number, the successor of a is a number.
  23^{α}. $\mathbf{N}_{\alpha'\to o}x_{\alpha'} \supset \mathbf{N}_{\alpha'\to o}(\mathbf{S}_{\alpha'\to\alpha'}x_{\alpha'})$
- 0 is not the successor of a number.
  25^{α}. $\mathbf{N}_{\alpha'\to o}x_{\alpha'} \supset \mathbf{S}_{\alpha'\to\alpha'}x_{\alpha'} \neq \mathbf{0}_{\alpha'}$
- Two numbers of which the successors are equal are themselves equal.
  26^{α}. $\mathbf{N}_{\alpha'\to o}x_{\alpha'} \supset \mathbf{N}_{\alpha'\to o}y_{\alpha'} \supset \mathbf{S}_{\alpha'\to\alpha'}x_{\alpha'} = \mathbf{S}_{\alpha'\to\alpha'}y_{\alpha'} \supset x_{\alpha'} = y_{\alpha'}$
- (induction axiom) If a set S of numbers contains 0 and also the successor of every number in S, then every number is in S.
  24^{α}. $f_{\alpha'\to o}\mathbf{0}_{\alpha'} \supset (\forall x_{\alpha'}.\mathbf{N}_{\alpha'\to o}x_{\alpha'} \supset f_{\alpha'\to o}x_{\alpha'} \supset f_{\alpha'\to o}(\mathbf{S}_{\alpha'\to\alpha'}x_{\alpha'})) \supset \mathbf{N}_{\alpha'\to o}x_{\alpha'} \supset f_{\alpha'\to o}x_{\alpha'}$

Induction Theorem

From 24^{α} and the deduction theorem VII, we have

VIII. If

1. $x_{\alpha'}$ is not a free variable of $A^1_o, A^2_o, \cdots, A^n_o, F_{\alpha' \to o}$,
2. $A^1_o, A^2_o, \cdots, A^n_o \vdash F_{\alpha' \to o}\mathbf{0}_{\alpha'}$, and
3. $A^1_o, A^2_o, \cdots, A^n_o, \mathbf{N}_{\alpha' \to o}x_{\alpha'}, F_{\alpha' \to o}x_{\alpha'} \vdash F_{\alpha' \to o}(\mathbf{S}_{\alpha' \to \alpha'}x_{\alpha'})$,

then $A^1_o, A^2_o, \cdots, A^n_o \vdash \mathbf{N}_{\alpha' \to o}x_{\alpha'} \supset F_{\alpha' \to o}x_{\alpha'}$.

PROOFS OF 25^α AND 26^α

- For proving 25^α,

25^α. $\mathbf{N}_{\alpha'\to o}x_{\alpha'} \supset \mathbf{S}_{\alpha'\to\alpha'}x_{\alpha'} \neq \mathbf{0}_{\alpha'}$

27^α. $\exists x_\alpha.\exists y_\alpha.x_\alpha \neq y_\alpha$

28^α. $\mathbf{S}_{\alpha'\to\alpha'}x_{\alpha'} \neq \mathbf{0}_{\alpha'}$

- For proving 26^α,

26^α. $\mathbf{N}_{\alpha'\to o}x_{\alpha'} \supset \mathbf{N}_{\alpha'\to o}y_{\alpha'} \supset \mathbf{S}_{\alpha'\to\alpha'}x_{\alpha'} = \mathbf{S}_{\alpha'\to\alpha'}y_{\alpha'} \supset x_{\alpha'} = y_{\alpha'}$
(only when α is either ι, ι', ι'', $\cdots$)

29^α. $\mathbf{N}_{\alpha''\to o} \supset \mathbf{N}_{\alpha'\to o}(n_{\alpha''}\mathbf{S}_{\alpha'\to\alpha'}\mathbf{0}_{\alpha'})$

30^α. $\mathbf{N}_{\alpha''\to o}m_{\alpha''} \supset \mathbf{N}_{\alpha''\to o}n_{\alpha''} \supset m_{\alpha''}\mathbf{S}_{\alpha'\to\alpha'}\mathbf{0}_{\alpha'} = n_{\alpha''}\mathbf{S}_{\alpha'\to\alpha'}\mathbf{0}_{\alpha'} \supset m_{\alpha''} = n_{\alpha''}$
(only when α is either ι, ι', ι'', $\cdots$)

Proof of 27^o

$$17^o\!: \; x = y \supset f_{o\to o}x \supset f_{o\to o}y$$

$\{M/x,\ \neg M/y,\ \lambda z.\neg(M \supset \neg z)/f_{o\to o}\}$
where $M = p \vee \neg p$

Rule IV'

$$(M = \neg M) \supset \neg(M \supset \neg M) \supset \neg(M \supset \neg\neg M)$$

Rule V — $(q \supset r \supset s) \supset (r \supset q \supset s)$

$$\neg(M \supset \neg M) \supset (M = \neg M) \supset \neg(M \supset \neg\neg M)$$

Rule V — $\neg(M \supset \neg M)$

$$(M = \neg M) \supset \neg(M \supset \neg\neg M)$$

$$(M \supset \neg\neg M) \supset (M \neq \neg M)$$

Rule V — $q \supset \neg\neg q$

$$M \neq \neg M$$

Rule III

$$(\lambda x.\lambda y.x \neq y)M(\neg M)$$

Rule V — $13^o : f_{o\to o}y \supset \exists x.f_{o\to o}x$

$$\exists x.\exists y.x \neq y$$

$$z_\alpha \neq t_\alpha \vdash \mathbf{K} z_\alpha x_\beta \neq \mathbf{K} t_\alpha x_\beta$$

$$17^\alpha\colon\ x_\alpha = y_\alpha \supset f_{\alpha\to o} x_\alpha \supset f_{\alpha\to o} y_\alpha$$

$$\left\{ \begin{array}{l} \mathbf{K} z_\alpha / x_\alpha, \mathbf{K} t_\alpha / y_\alpha, \\ \lambda w_{\beta\to\alpha}. w_{\beta\to\alpha} x_\beta \neq \mathbf{K} t_\alpha x_\beta / f_{\alpha\to o} \end{array} \right\}$$

$$\mathbf{K} t_\alpha = \mathbf{K} z_\alpha \supset \mathbf{K} z_\alpha x_\beta \neq \mathbf{K} t_\alpha x_\beta \supset \mathbf{K} t_\alpha x_\beta \neq \mathbf{K} t_\alpha x_\beta$$

$$z_\alpha \neq t_\alpha, \mathbf{K} t_\alpha = \mathbf{K} z_\alpha \vdash \mathbf{K} t_\alpha x_\beta \neq \mathbf{K} t_\alpha x_\beta$$

$$z_\alpha \neq t_\alpha \vdash \mathbf{K} t_\alpha = \mathbf{K} z_\alpha \supset \mathbf{K} t_\alpha x_\beta \neq \mathbf{K} t_\alpha x_\beta$$

$$z_\alpha \neq t_\alpha \vdash \mathbf{K} t_\alpha x_\beta = \mathbf{K} t_\alpha x_\beta \supset \mathbf{K} t_\alpha \neq \mathbf{K} z_\alpha$$

$$z_\alpha \neq t_\alpha \vdash \mathbf{K} t_\alpha \neq \mathbf{K} z_\alpha$$

$$z_\alpha \neq t_\alpha \supset \exists f_{\beta\to\alpha}. \exists g_{\beta\to\alpha}. f \neq g$$

$$(\exists z_\alpha. \exists t_\alpha. z_\alpha \neq t_\alpha) \supset \exists f_{\beta\to\alpha}. \exists g_{\beta\to\alpha}. f \neq g$$

$$\exists f_{\beta\to\alpha}. \exists g_{\beta\to\alpha}. f \neq g$$

Proof of 27$^{b\ !\ a}$ under 27^{a}

$$z_\alpha \neq t_\alpha \vdash \mathbf{S}_{\alpha'\to\alpha'} x_{\alpha'} (\mathbf{K}_{\alpha\to\alpha\to\alpha} z_\alpha) t_\alpha \neq \mathbf{0}_{\alpha'} (\mathbf{K}_{\alpha\to\alpha\to\alpha} z_\alpha) t_\alpha$$

$$17^\alpha\text{:}\ x_\alpha = y_\alpha \supset f_{\alpha\to o} x_\alpha \supset f_{\alpha\to o} y_\alpha$$

$$\begin{aligned}\mathbf{S}_{\alpha'\to\alpha'} x_{\alpha'} = \mathbf{0}_{\alpha'} \ \vdash\ & \mathbf{S}_{\alpha'\to\alpha'} x_{\alpha'} (\mathbf{K}_{\alpha\to\alpha\to\alpha} z_\alpha) t_\alpha \neq \mathbf{0}_{\alpha'} (\mathbf{K}_{\alpha\to\alpha\to\alpha} z_\alpha) t_\alpha \\ \supset\ & \mathbf{0}_{\alpha'} (\mathbf{K}_{\alpha\to\alpha\to\alpha} z_\alpha) t_\alpha \neq \mathbf{0}_{\alpha'} (\mathbf{K}_{\alpha\to\alpha\to\alpha} z_\alpha) t_\alpha\end{aligned}$$

$$z_\alpha \neq t_\alpha \vdash \mathbf{S}_{\alpha'\to\alpha'} x_{\alpha'} = \mathbf{0}_{\alpha'} \supset \mathbf{0}_{\alpha'} (\mathbf{K}_{\alpha\to\alpha\to\alpha} z_\alpha) t_\alpha \neq \mathbf{0}_{\alpha'} (\mathbf{K}_{\alpha\to\alpha\to\alpha} z_\alpha) t_\alpha$$
$$z_\alpha \neq t_\alpha \vdash \mathbf{0}_{\alpha'} (\mathbf{K}_{\alpha\to\alpha\to\alpha} z_\alpha) t_\alpha = \mathbf{0}_{\alpha'} (\mathbf{K}_{\alpha\to\alpha\to\alpha} z_\alpha) t_\alpha \supset \mathbf{S}_{\alpha'\to\alpha'} x_{\alpha'} \neq \mathbf{0}_{\alpha'}$$
$$z_\alpha \neq t_\alpha \vdash \mathbf{S}_{\alpha'\to\alpha'} x_{\alpha'} \neq \mathbf{0}_{\alpha'}$$

$$\exists z_\alpha . \exists t_\alpha . z_\alpha \neq t_\alpha \supset \mathbf{S}_{\alpha'\to\alpha'} x_{\alpha'} \neq \mathbf{0}_{\alpha'}$$

$$\mathbf{S}_{\alpha'\to\alpha'} x_{\alpha'} \neq \mathbf{0}_{\alpha'}$$

Proof of 28^{a} and 25^{a}

PROPERTIES OF $\mathbf{T}_{\alpha' \to \alpha''}$

- Theorems

31^{α}. $\mathbf{N}_{\alpha' \to o} x_{\alpha'} \supset \mathbf{N}_{\alpha'' \to o}(\mathbf{T}_{\alpha' \to \alpha''} x_{\alpha'})$

32^{α}. $\mathbf{N}_{\alpha' \to o} x_{\alpha'} \supset \mathbf{T}_{\alpha' \to \alpha''} x_{\alpha'} \mathbf{S}_{\alpha' \to \alpha'} \mathbf{0}_{\alpha'} = x_{\alpha'}$

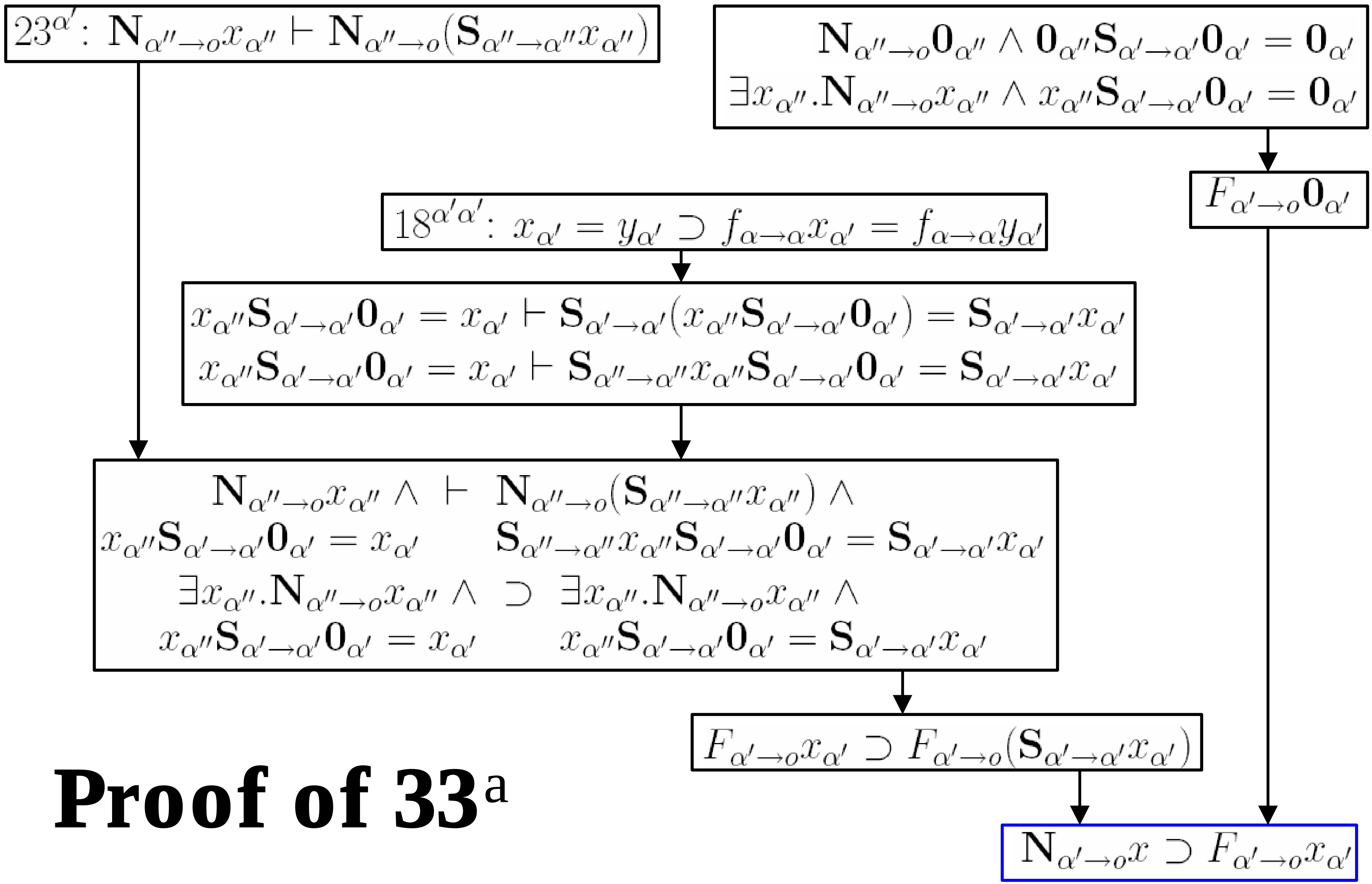
33^{α}. $\mathbf{N}_{\alpha' \to o} x_{\alpha'} \supset \exists x_{\alpha''}. \mathbf{N}_{\alpha'' \to o} x_{\alpha''} \wedge x_{\alpha''} \mathbf{S}_{\alpha' \to \alpha'} \mathbf{0}_{\alpha'} = x_{\alpha'}$

34^{α}. $\mathbf{N}_{\alpha' \to o} x_{\alpha'} \supset \mathbf{T}_{\alpha' \to \alpha''}(\mathbf{S}_{\alpha' \to \alpha'} x_{\alpha'}) = \mathbf{S}_{\alpha'' \to \alpha''}(\mathbf{T}_{\alpha' \to \alpha''} x_{\alpha'})$

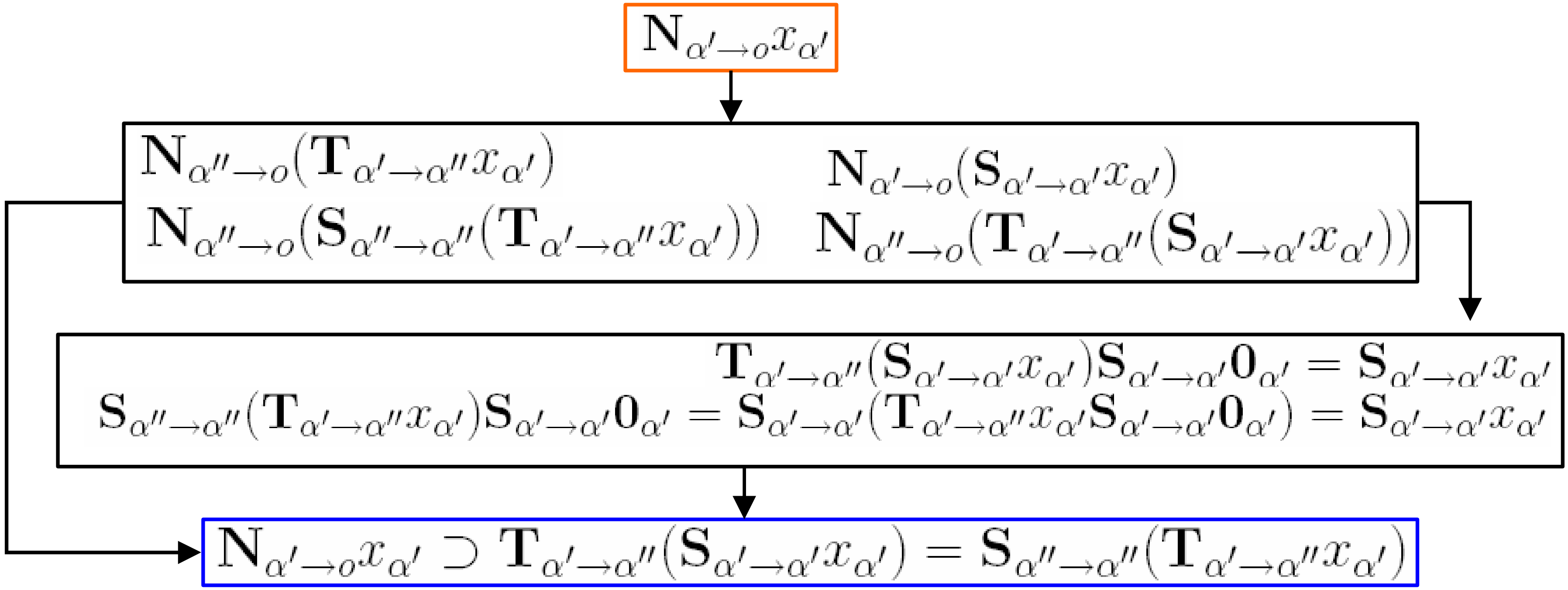
35^{α}. $\mathbf{T}_{\alpha' \to \alpha''} \mathbf{0}_{\alpha'} = \mathbf{0}_{\alpha''}$

(31,32,34,35 only when α is either $\iota, \iota', \iota'', \cdots$)

$$F_{\alpha'\to o} = \lambda w_{\alpha'}.\exists x_{\alpha''}.\mathbf{N}_{\alpha''\to o}x_{\alpha''} \wedge x_{\alpha''}\mathbf{S}_{\alpha'\to\alpha'}\mathbf{0}_{\alpha'} = w_{\alpha'}$$



Proof of 33[a]



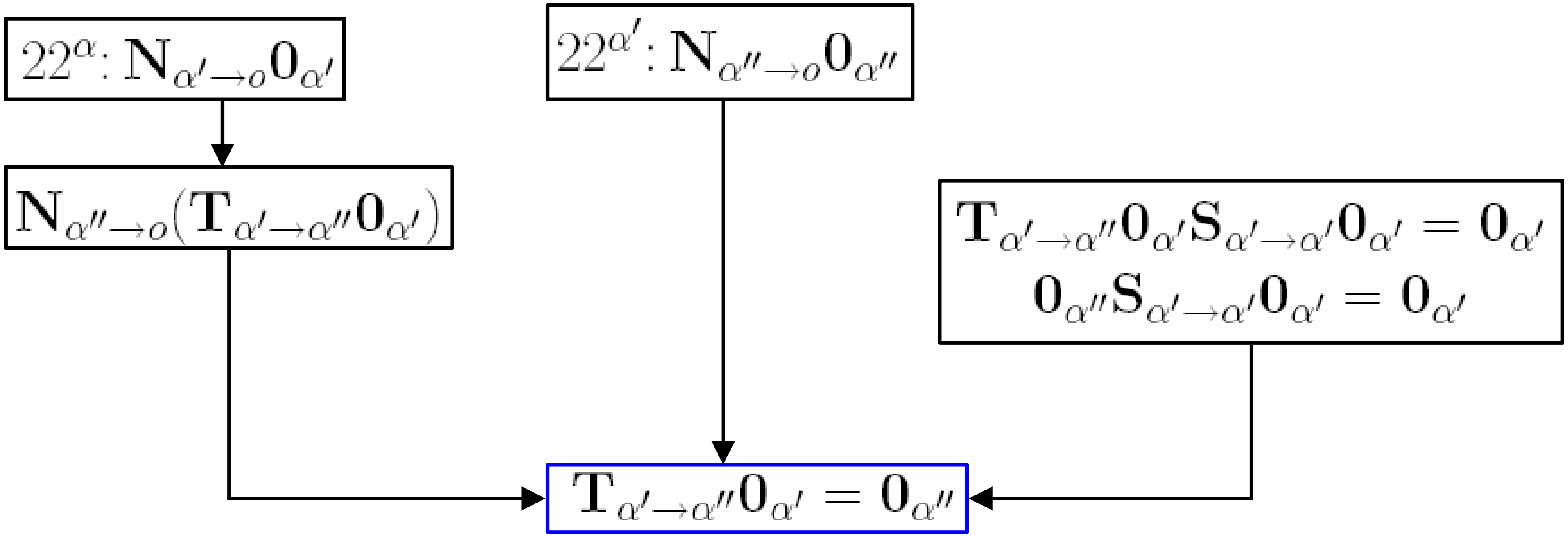
Proof of 34^{a}

23^{α}: $\mathbf{N}_{\alpha'\to o} x_{\alpha'} \supset \mathbf{N}_{\alpha'\to o}(\mathbf{S}_{\alpha'\to\alpha'} x_{\alpha'})$

31^{α}: $\mathbf{N}_{\alpha'\to o} x_{\alpha'} \supset \mathbf{N}_{\alpha''\to o}(\mathbf{T}_{\alpha'\to\alpha''} x_{\alpha'})$

32^{α}: $\mathbf{N}_{\alpha'\to o} x_{\alpha'} \supset \mathbf{T}_{\alpha'\to\alpha''} x_{\alpha'}\mathbf{S}_{\alpha'\to\alpha'}\mathbf{0}_{\alpha'} = x_{\alpha'}$

30^{α}: $\mathbf{N}_{\alpha''\to o} m_{\alpha''} \supset \mathbf{N}_{\alpha''\to o} n_{\alpha''} \supset m_{\alpha''}\mathbf{S}_{\alpha'\to\alpha'}\mathbf{0}_{\alpha'} = n_{\alpha''}\mathbf{S}_{\alpha'\to\alpha'}\mathbf{0}_{\alpha'} \supset m_{\alpha''} = n_{\alpha''}$



Proof of 35^{a}

31^{α}: $\mathbf{N}_{\alpha'\to o}x_{\alpha'} \supset \mathbf{N}_{\alpha''\to o}(\mathbf{T}_{\alpha'\to\alpha''}x_{\alpha'})$

32^{α}: $\mathbf{N}_{\alpha'\to o}x_{\alpha'} \supset \mathbf{T}_{\alpha'\to\alpha''}x_{\alpha'}\mathbf{S}_{\alpha'\to\alpha'}\mathbf{0}_{\alpha'} = x_{\alpha'}$

30^{α}: $\mathbf{N}_{\alpha''\to o}m_{\alpha''} \supset \mathbf{N}_{\alpha''\to o}n_{\alpha''} \supset m_{\alpha''}\mathbf{S}_{\alpha'\to\alpha'}\mathbf{0}_{\alpha'} = n_{\alpha''}\mathbf{S}_{\alpha'\to\alpha'}\mathbf{0}_{\alpha'} \supset m_{\alpha''} = n_{\alpha''}$

PRIMITIVE RECURSION

- Given formulas $A_{\alpha'}$ and $B_{\alpha' \to \alpha' \to \alpha'}$, there exists $F_{\alpha' \to \alpha'}$ such that

$$\begin{aligned} F_{\alpha' \to \alpha'} \mathbf{0}_{\alpha'} &= A_{\alpha'} \\ \mathbf{N}_{\alpha' \to o} x_{\alpha'} \supset F_{\alpha' \to \alpha'}(\mathbf{S}_{\alpha' \to \alpha'} x_{\alpha'}) &= B_{\alpha' \to \alpha' \to \alpha'} x_{\alpha'} (F_{\alpha' \to \alpha'} x_{\alpha'}) \end{aligned}$$

  where $x_{\alpha'}$ is not a free variable of $A_{\alpha'}$, $B_{\alpha' \to \alpha' \to \alpha'}$, or $F_{\alpha' \to \alpha'}$.

- $F_{\alpha' \to \alpha'}$ is:

$$\begin{array}{l} \lambda x_{\alpha'} . \mathbf{T}_{\alpha'' \to \alpha'''}(\mathbf{T}_{\alpha' \to \alpha''} x_{\alpha'}) \\ \left(\lambda y_{\alpha''} . \left\langle \begin{array}{l} \mathbf{S}_{\alpha' \to \alpha'}(y_{\alpha''}(\mathbf{K}_{\alpha' \to \alpha' \to \alpha'} \mathbf{I}_{\alpha' \to \alpha'}) \mathbf{0}_{\alpha'}), \\ B_{\alpha' \to \alpha' \to \alpha'}(y_{\alpha''}(\mathbf{K}_{\alpha' \to \alpha' \to \alpha'} \mathbf{I}_{\alpha' \to \alpha'}) \mathbf{0}_{\alpha'})(y_{\alpha''}(\mathbf{K}_{\alpha' \to \alpha' \to \alpha'} \mathbf{0}_{\alpha'}) \mathbf{I}_{\alpha' \to \alpha'}) \end{array} \right\rangle \right) \\ \langle \mathbf{0}_{\alpha'}, A_{\alpha'} \rangle \, (\mathbf{K}_{\alpha' \to \alpha' \to \alpha'} \mathbf{0}_{\alpha'}) \end{array}$$

- $\mathbf{P}_{\alpha' \to \alpha'}$ is a primitive recursive function for the case that $A_{\alpha'} = \mathbf{0}_{\alpha'}$ and $B_{\alpha' \to \alpha' \to \alpha'} = \lambda y_{\alpha'} . \lambda z_{\alpha'} . y_{\alpha'}$:

PROOF FOR $\mathbf{P}_{\alpha'\to\alpha'}$

$$\mathbf{P}_{\alpha'''\to\alpha'} \to \lambda n_{\alpha'''}.n_{\alpha'''} \left(\lambda p_{\alpha''}.\left\langle \begin{array}{r} \mathbf{S}_{\alpha'\to\alpha'}(p_{\alpha''}(\mathbf{K}_{\alpha'\to\alpha'\to\alpha'}\mathbf{I}_{\alpha'\to\alpha'})\mathbf{0}_{\alpha'}), \\ p_{\alpha''}(\mathbf{K}_{\alpha'\to\alpha'\to\alpha'}\mathbf{I}_{\alpha'\to\alpha'})\mathbf{0}_{\alpha'} \end{array} \right\rangle\right) \langle \mathbf{0}_{\alpha'}, \mathbf{0}_{\alpha'}\rangle \, (\mathbf{K}_{\alpha'\to\alpha'\to\alpha'}\mathbf{0}_{\alpha'}) \, \mathbf{I}_{\alpha'\to\alpha'}$$

36^α. $\mathbf{N}_{\alpha'\to o}n_{\alpha'} \supset \lambda f_{\alpha\to\alpha}\lambda x_\alpha(n_{\alpha'}f_{\alpha\to\alpha}x_\alpha) = n_{\alpha'}$

37^α. $\mathbf{N}_{\alpha'\to o}m_{\alpha'} \supset \mathbf{N}_{\alpha'\to o}n_{\alpha'} \supset \langle m_{\alpha'}, n_{\alpha'}\rangle \, (\mathbf{K}_{\alpha'\to\alpha'\to\alpha'}\mathbf{I}_{\alpha'\to\alpha'})\mathbf{0}_{\alpha'} = m_{\alpha'}$

38^α. $\mathbf{N}_{\alpha'\to o}m_{\alpha'} \supset \mathbf{N}_{\alpha'\to o}n_{\alpha'} \supset \langle m_{\alpha'}, n_{\alpha'}\rangle \, (\mathbf{K}_{\alpha'\to\alpha'\to\alpha'}\mathbf{0}_{\alpha'})\mathbf{I}_{\alpha\to\alpha} = n_{\alpha'}$

39^α. $\mathbf{N}_{\alpha'''\to o}n_{\alpha'''} \supset n_{\alpha'''}\left(\lambda p_{\alpha''}.\left\langle \begin{array}{r} \mathbf{S}_{\alpha'\to\alpha'}(p_{\alpha''}(\mathbf{K}_{\alpha'\to\alpha'\to\alpha'}\mathbf{I}_{\alpha'\to\alpha'})\mathbf{0}_{\alpha'}), \\ p_{\alpha''}(\mathbf{K}_{\alpha'\to\alpha'\to\alpha'}\mathbf{I}_{\alpha'\to\alpha'})\mathbf{0}_{\alpha'} \end{array} \right\rangle\right) \langle \mathbf{0}_{\alpha'}, \mathbf{0}_{\alpha'}\rangle \, (\mathbf{K}_{\alpha'\to\alpha'\to\alpha'}\mathbf{I}_{\alpha'\to\alpha'})\mathbf{0}_{\alpha'} = n_{\alpha'}\alpha''\mathbf{S}_{\alpha''\to\alpha''}\mathbf{0}_{\alpha''}\mathbf{S}_{\alpha'\to\alpha'}\mathbf{0}_{\alpha'}$

40^α. $\mathbf{N}_{\alpha'\to o}\alpha''n_{\alpha'''} \supset \mathbf{P}_{\alpha'\to\alpha'''}(\alpha''n_{\alpha'''}) = n_{\alpha'}\alpha''\mathbf{S}_{\alpha''\to\alpha''}\mathbf{0}_{\alpha''}\mathbf{S}_{\alpha'\to\alpha'}\mathbf{0}_{\alpha'}$

41^α. $\mathbf{P}_{\alpha'\to\alpha'''}\mathbf{0}_{\alpha'''} = \mathbf{0}_{\alpha'}$

42^α. $\mathbf{P}_{\alpha'\to\alpha'}\mathbf{0}_{\alpha'} = \mathbf{0}_{\alpha'}$

43^α. $\mathbf{N}_{\alpha'\to o}n_{\alpha'} \supset \mathbf{P}_{\alpha'\to\alpha'}(\mathbf{S}_{\alpha'\to\alpha'}n_{\alpha'}) = n_{\alpha'}$

SUMMARY

- A typed higher-order logic system which incorporates λ-calculus.
  - Peano's arithmetic
  - Primitive recursion
- A basis formalizm in modern higher-order theorem provers.