Algebraic Tools for Modal Logic

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Contents

Syntax and Semantics

Well-formed formulas φ of the *basic* modal language

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \psi \vee \varphi \mid \Diamond\varphi$$

여기서 p는 proposition letter, $\bot$, $\neg$, $\vee$은 각각 arity 0,1,2인 connective, $\Diamond$는 unary modal operator이며, 이것의 dual은 $\Box\varphi := \neg\Diamond\neg\varphi$로 정의하여 사용한다.

일반적인 modal language에서는 임의의 유한한 arity를 가지는 modal operator를 여럿 가질 수 있다.

Syntax and Semantics, continued

A *frame* for the basic modal language is a pair $F = (W, R)$ such that W is a nonempty set and R is a binary relation on W.

A *model* for the basic modal language is a pair $M = (F, V)$ where F is a frame for the basic modal language and V is a function assigining to each proposition letter p a subset $V(p)$ of W. V is called a *valuation*.

We write $M, w \Vdash \varphi$ in case the formula φ is *satisfied* (or *true*) in model M at *state* w. $\Vdash$는 재귀적으로 정의되며, 여러 경우 중 가장 중요한 하나만 아래에 보였다.

$$M, w \Vdash \Diamond\varphi \text{ iff } (\exists v \in W)(Rwv \text{ and } M, v \Vdash \varphi)$$

φ is *globally true* in M iff it is true in M at all points in W.

Syntax and Semantics, continued

A formula φ is *valid at a state w on a frame F* (notation: $F, w \Vdash \varphi$) iff φ is true at w in every model based on F.

A formula φ is *valid in a frame F* (notation: $F \Vdash \varphi$) iff φ is valid at every state on F.

A formula φ is *valid on a class of frames $\mathbb{F}$* (notation: $\mathbb{F} \Vdash \varphi$) iff it is valid on every frame $F \in \mathbb{F}$.

A formula is *valid* iff it is valid in the class of all frames.

The set of all formulas valid in a class of frames $\mathbb{F}$ is called the *logic* of $\mathbb{F}$, and denoted by $\Lambda_{\mathbb{F}}$.

Syntax and Semantics, continued

models $<$ *general frames* $<$ frames

A *general frame* is a pair (F, A) where $A \subseteq \mathcal{P}(W)$ satisfies the following closure conditions:

1. if $X, Y \in A$, then $X \cup Y \in A$,
2. if $X \in A$, then $W - X \in A$,
3. if $X \in A$, then $\{w \in W \mid Rwx \text{ for some } x \in X\} \in A$.

A *model based on a general frame* (F, A) is a triple (F, A, V) where $V(p) \in A$ for each proposition letter p.

The validity of a formula in a general frame is defined in the obvious way.

Completeness

A *(normal modal) logic* in the basic modal language is a set Λ of modal formulas that contains, besides all propositional tautologies, the axioms

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

while it is closed under the inference rules

- (modus ponens) if $\varphi \to \psi \in \Lambda$ and $\varphi \in \Lambda$ then $\psi \in \Lambda$,
- (necessitation) if $\varphi \in \Lambda$ then $\Box\varphi \in \Lambda$.

Usually we require *consistency* for a set of formulas to be a logic.

If Λ is a logic, then we may write $\vdash_\Lambda \varphi$ in place of $\varphi \in \Lambda$.

Completeness, continued

A logic Λ is *sound* with respect to a class $\mathbb{F}$ of frames (or models or general frames) iff $\Lambda \subseteq \Lambda_{\mathbb{F}}$, and *complete* iff $\Lambda \supseteq \Lambda_{\mathbb{F}}$

Proving soundeness is often easy, but proving completeness isn't.

A logic Λ is *complete* iff $\Lambda = \Lambda_{\mathbb{F}}$ for some frame class $\mathbb{F}$.

Canonical Models

Completeness $\Lambda \supseteq \Lambda_{\mathbb{F}}$를 증명하기 위하여는 모델을 만들어 내야 한다.

즉 $\varphi \notin \Lambda$이면 φ가 어떤 모델 M과 state w에서 $M, w \not\Vdash \varphi$ 임을 보여야 한다. 다시 말하면 $M, w \Vdash \neg\varphi$ 인 모델 M(그리고 state w)를 만들어야 한다.

For a set $\Gamma \cup \{\varphi\}$ of formulas, we say φ is *deducible in* Λ *from* Γ if there exists $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\vdash_\Lambda \psi_1 \wedge \cdots \wedge \psi_n \to \varphi$. In this case we write $\Gamma \vdash_\Lambda \varphi$.

Γ is Λ-*inconsistent* iff $\Gamma \vdash_\Lambda \perp$, and Λ-*consistent* otherwise. Γ is *maximal* Λ-*consistent* iff Γ is consistent and every proper superset of Γ is Λ-inconsistent. In this case we say Γ is a Λ-*MCS*.

Canonical Models, continued

If Γ is a Λ-MCS, then

- Γ is closed under modus-ponens.
- $\Lambda \subseteq \Gamma$.
- for all φ, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.
- for all φ and ψ, $\varphi \vee \psi \in \Gamma$ iff ($\varphi \in \Gamma$ or $\psi \in \Gamma$).

Lindenbaum's Lemma: Each Λ-consistent set of formulas can be extended to a Λ-MCS.

Canonical Models, continued

The *canonical model* M^Λ is the triple $(W^\Lambda, R^\Lambda, V^\Lambda)$ where

- W^Λ is the set of all Λ-MCS's.
- the *canonical relataion* R^Λ is the binary relation on W^Λ defined by
$$R^\Lambda wu \ \Leftrightarrow \ (\forall\psi)(\psi \in u \to \Diamond\psi \in w).$$
- the *canonical valuation* V^Λ is defined by
$$V^\Lambda(p) = \{w \in W^\Lambda \mid p \in w\}.$$

The pair (W^Λ, R^Λ) is called the *canonical frame* for the logic Λ.

Canonical Models, continued

Above definition says, that for a proposition letter p,

$$(p \text{ is true at state } w) \text{ iff } (p \in w).$$

We want to lift this *"truth=membership"* to arbitrary formulas.

The proof is done by induction on the complexity of φ. For boolean case, it is easy. For modality,

$$\begin{aligned} M^\Lambda, w \Vdash \Diamond\varphi &\Leftrightarrow \exists v(R^\Lambda wv \wedge M^\Lambda, v \Vdash \varphi) \\ &\Leftrightarrow \exists v(R^\Lambda wv \wedge \varphi \in v) \\ &\Rightarrow \Diamond\varphi \in w. \end{aligned}$$

The converse '$\Leftarrow$' holds by the *Existence Lemma*, whose prooof is not given here.

Canonical Models, continued

Proposition: A logic Λ is complete w.r.t. a class $\mathbb{F}$ of frames (or models of general frames) iff every Λ-consistent formula is satisfiable in some $F \in \mathbb{F}$.

Canonical Model Theorem: For any normal modal logic Λ and for every formula φ, we have

$$M^{\Lambda} \Vdash \varphi \;\Leftrightarrow \vdash_{\Lambda} \varphi.$$

proof: Soundeness is easier. For completeness, suppose φ is Λ-consistent. Then let Σ be a Λ-MCS that has φ as an element. Thus $M^{\Lambda}, \Sigma \Vdash \varphi$, whence φ is satisfiable in the canonical model. Now the proof is complete by applying the previous proposition.

We can obtain many important completenss results using the canonical models.

Canonicity

A formula φ is *canonical* if, for every normal logic Λ, $\varphi \in \Lambda$ implies that φ is valid on the canonical frame for Λ.

A normal logic Λ is *canonical* if each of its members is valid on the canonical frame for Λ.

Theorem: Every canonical logic is complete.

Canonicity, continued

We are interested in the following:

- When is a formula canonical? (This is an undecidable problem. Nevertheless there is an important class of canonical formulas called the *Sahlqvist formulas*.)
- When is a canonical formula "elementary" (i.e., first-order)?
- How do we show that a formula is canonical? (Algebraic methods are available.)
- Given two sets of axioms, how do we show that they axiomatize the same (or different) logic? (It is almost always the case that semantic methods are better than syntactic ones.)

Logic as Algebras