모달 로직의 소개

LiComR Summer Workshop 2003

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월드컵 4강, 그 다음 해 여름

Modal logic

 얘기할 순서

• 모달 로직이란?
• 간단한 모달 언어와 그 시멘틱스
• 일반 모달 언어와 그 시멘틱스
• 공리계와 그 확장
• 그리고 아무도 ...
모달 로직에 대한 말 말 말?
양상 논리(模相論理) - 양상, 양식, modality 를 다루는 논리 체계

Propositional logic + modal operators

C.I. Lewis
A Description Language for Relational structures

'....이 필연적이다', '....이 가능하다'와 같은 표현에 대한 연역 활동을
공부하는 분야

ألم(belief), 시간(tense), deontic(윤리) 등에 관한 논리로서
철학적인 논의의 형식적인 분석 뿐 아니라 컴퓨터 과학 등에 적용된
data.

<p>| | |</p>
<table>
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| **Alethic**   | □ : it is necessary that ...
|               | ◊ : it is possible that ... |
| **Deontic**   | O : it is obligatory that ...
|               | P : it is permitted that ...
|               | F : it is forbidden that ... |
| **Temporal**  | G : it will be always true in the future that ...
|               | F : it will be true at some point in the future that ...
|               | H : it was always true in the past that ...
|               | P : it was true at some point in the past that ...
| **Doxastic**  | B_x : x believes that ...
| **Epistemic** | K_x : x knows that ...
| **Provability** | P : it is provable that ... |
Basic modal logic

Alphabet:

- A set of propositional letters \( p, q, \ldots \)
- Propositional connectives \( \neg \) and \( \land \) and constant true \( \top \) (\( \lor \), \( \rightarrow \), \( \leftrightarrow \), \( \bot \) are definable)
- modality \( \Box \) (\( \Diamond \) is definable)

Well-formed formula \( \phi, \psi \):

\[
p \mid \top \mid \neg \phi \mid \phi \land \psi \mid \Box \phi
\]
Modal logic

"Relational structures"

A *frame* for the basic modal language $\mathcal{F} = (W, R)$ where $W$ is a non-empty set of possible worlds and $R$ is a binary relation on $W$.

A *model* for the basic modal language $\mathcal{M} = (\mathcal{F}, V)$ where $\mathcal{F}$ is a frame and $V$ is a valuation function $\Phi \to \mathcal{P}(W)$.

"Satisfaction"

$\mathcal{M}, w \models p$ if $w \in V(p)$, where $p \in \Phi$

$\mathcal{M}, w \models \top$ if always

$\mathcal{M}, w \models \neg \phi$ if $\mathcal{M}, w \not\models \phi$

$\mathcal{M}, w \models \phi \land \psi$ if $\mathcal{M}, w \models \phi$ and $\mathcal{M}, w \models \psi$

$\mathcal{M}, w \models \Box \phi$ if for every $v \in W$ such that $Rwv$, we have $\mathcal{M}, v \models \phi$

For a set $\Sigma$ of formulas

$\mathcal{M}, w \models \Sigma$ if all members of $\Sigma$ are true at $w$

For the valuation of arbitrary formulas

$V(\phi) = \{w \mid \mathcal{M}, w \models \phi\}$
Modal logic

"Global truth and satisfiability"

A formula $\phi$ is globally or universally true in a model $M$

$M \models \phi$ if $M, w \models \phi$ for all $w \in W$

A formula $\phi$ is satisfiable in a model $M$

$\exists w. M, w \models \phi$

A set $\Sigma$ of formulas is globally true in a model $M$

$M \models \Sigma$ if $M, w \models \Sigma$ for all $w \in W$

A set $\Sigma$ of formulas is satisfiable in a model $M$

$\exists w. M, w \models \Sigma$

Example 1

$\mathcal{F} = (\{w_1, w_2, w_3, w_4, w_5\}, R)$ where $Rw_i w_j$ iff $j = i + 1$:

$V(p) = \{w_2, w_3\}$

$V(q) = \{w_1, w_2, w_3, w_4, w_5\}$

$V(r) = \emptyset$

$M, w_1 \models \lozenge \Box p$

$M, w_1 \not\models \lozenge \Box p \rightarrow p$

$M, w_2 \models \lozenge (p \land \lnot r)$

$M, w_1 \models q \land \lozenge (q \land \lozenge (q \land \lozenge (q \land \lozenge q)))$
Modal logic

"예제 2"

$$
\mathfrak{F} = (\{1, 2, 3, 4, 6, 8, 12, 24\}, R) \text{ where } Rxy \iff x \neq y \text{ and } x \text{ divides } y:
$$

$$
V(p) = \{4, 8, 12, 24\}, \quad V(q) = \{6\}
$$

$$
\mathcal{M}, 6 \models \Box p
\quad \mathcal{M}, 2 \not\models \Box p
\quad \mathcal{M}, 2 \models \Diamond (q \land \Box p) \land \Diamond (\neg q \land \Box p)
$$

" $R$-accessible Possible worlds"

$$
\mathfrak{F} = (W, R)
\quad \mathcal{M} = (\mathfrak{F}, V)
$$

When $R = W \times W$, $(W, R, V), w \models \Box \phi$ if $\forall v \in W. (W, R, V), v \models \phi$

$$
:\:
:\:
$$
When $R = \emptyset$, $(W, R, V), w \models \Box \phi$ if always
Modal logic

"Modal similarity type"

\[ \tau = (O, \rho) \]

where \( O \) is a non-empty set of \textit{modal operators} \( \triangle, \triangle_0, \triangle_1, \ldots \)
and \( \rho \) is an arity function \( O \to \mathbb{N} \).

For multi-modality

\[ \Box_a \text{ or } [a] \]
\[ \Diamond_a \text{ or } \langle a \rangle \]
where \( a \) is taken from some index set.

Modal logic

"Modal language"

\[ ML(\tau, \Phi) \]

where \( \tau = (O, \rho) \) is a modal similarity type
and \( \Phi \) is a set of propositional letters.

The set \( \text{Form}(\tau, \Phi) \) of modal formulas over \( \tau \) and \( \Phi \) is given by the rule

\[ \phi ::= p \mid \bot \mid \neg \phi \mid \phi_1 \land \phi_2 \mid \triangle(\phi_1, \ldots, \phi_{\rho(\triangle)}) \]

where \( p \) ranges over elements of \( \Phi \).

For each \( \triangle \in O \), the dual \( \nabla \) of \( \triangle \) is defined as

\[ \nabla(\phi_1, \ldots, \phi_n) = \neg \triangle(\neg \phi_1, \ldots, \neg \phi_n) \]
"τ-frame and τ-model"

\(\tau\)-frame \(\mathcal{F} = (W, R_\triangle)_{\triangle \in \tau}\) or \(\mathcal{F} = (W, \{R_\triangle \mid \triangle \in \tau\})\)

(i) a non-empty set \(W\) of possible worlds

(ii) for each \(n \geq 0\), and each \(n\)-ary modal operator \(\triangle\) in the similarity type \(\tau\), an \((n + 1)\)-ary relation \(R_\triangle\)

\(\tau\)-model \(\mathcal{M} = (\mathcal{F}, V)\)

(i) a \(\tau\)-frame \(\mathcal{F}\)

(ii) a valuation function \(V : \Phi \to \mathcal{P}(W)\)

---

"Satisfaction again"

\[\mathcal{M}, w \models p \text{ if } w \in V(p), \text{ where } p \in \Phi\]

\[\mathcal{M}, w \models \top \text{ if } \text{always}\]

\[\mathcal{M}, w \models \neg \phi \text{ if } \mathcal{M}, w \not\models \phi\]

\[\mathcal{M}, w \models \phi \land \psi \text{ if } \mathcal{M}, w \models \phi \text{ and } \mathcal{M}, w \models \psi\]

\[\mathcal{M}, w \models \triangle(\phi_1, \ldots, \phi_n) \text{ if for every } v_1, \ldots, v_n \in W \text{ such that } R_\triangle w v_1 \ldots v_n, \]

\[\text{we have, for each } i, \mathcal{M}, v_i \models \phi_i\]

**Note:** when \(\rho(\triangle) = 0\)

\[\mathcal{M}, w \models \triangle \text{ if } w \in R_\triangle\]
"예제 3"

similarity type \( \tau = (\{\triangle, \bigcirc\}, \{\triangle \mapsto 2, \bigcirc \mapsto 3\}) \)

\( \tau \)-frame \( \mathfrak{F} = (\{u, v, w, s\}, R_{\triangle}, S_{\bigcirc}) \)

\( R_{\triangle} = \{(u, v, w)\} \quad S_{\bigcirc} = \{(u, v, w, s)\} \)

\( V(p_0) = \{v\} \quad V(p_1) = \{w\} \quad V(p_2) = \{s\} \)

\[ \mathcal{M}, u \models \triangle(p_0, p_1) \rightarrow \bigcirc(p_0, p_1, p_2) \]

\[ \mathcal{M} \models \triangle(p_0, p_1) \rightarrow \bigcirc(p_0, p_1, p_2) \]

"Validity"

\[ \mathfrak{F}, w \models \phi \quad \text{iff} \quad \text{for every } \mathcal{M} \text{ such that } \mathcal{M} = (\mathfrak{F}, V) \]

\[ \mathcal{M}, w \models \phi \]

\[ \mathfrak{F} \models \phi \quad \text{iff} \quad \text{for every } w \in W \quad \mathfrak{F}, w \models \phi \]

\[ F \models \phi \quad \text{iff} \quad \text{for every } \mathfrak{F} \text{ in a class of frames } F \]

\[ \mathfrak{F} \models \phi \]

\[ \models \phi \quad \text{iff} \quad \text{for every } \mathfrak{F} \quad \mathfrak{F} \models \phi \]

\[ \phi \lor \psi \text{ is true at } w = \text{either } \phi \text{ or } \psi \text{ is true at } w \]

\[ \phi \lor \psi \text{ is valid on } \mathfrak{F} \neq \text{either } \phi \text{ or } \psi \text{ is valid on } \mathfrak{F} \]
"예제 4: \( \Diamond (p \lor q) \to (\Diamond p \lor \Diamond q) \) is valid on all frames?"

임의의 \( \mathfrak{F} , w , V \) 를 취하고 \( \mathfrak{F} , V , w \models \Diamond (p \lor q) \) 임을 가정하자. 그러면 정의에 의해서 \( Rwv \) 이고 \( \mathfrak{F} , V , v \models p \lor q \) 인 \( v \) 가 존재한다. 그런데 \( \mathfrak{F} , V , v \models p \lor q \) 이면 \( \mathfrak{F} , V , v \models p \) 이거나 \( \mathfrak{F} , V , v \models q \) 이다. 따라서 \( \mathfrak{F} , V , w \models \Diamond p \) 이거나 \( \mathfrak{F} , V , w \models \Diamond q \) 이고 어느 쪽이든 \( \mathfrak{F} , V , w \models \Diamond p \lor \Diamond q \) 이다.

"예제 5: \( \Box p \to \Box \Box p \) is not valid on all frames?"

\( \mathfrak{F} = \{0,1,2\}, \{(0,1),(1,2)\} \) 와 \( V(p) = 1 \) 을 취하면 counter-example을 만들 수 있다. 즉, \( \mathfrak{F} , V , 0 \not\models \Box p \) 이지만 \( \mathfrak{F} , V , 0 \models \Box \Box p \) 는 아니다.

"예제 6: \( \Box p \to \Box \Box p \) is valid on transitive frames?"

임의의 transitive \( \mathfrak{F} , w , V \) 를 취하고 \( \mathfrak{F} , V , w \models \Box p \) 임을 가정하자. 그러면 정의에 의해서 \( Rwu \) 인 모든 \( u \) 에 대하여 \( \mathfrak{F} , V , u \models p \) 이다. 그러면 \( R \) 이 transitive 이므로 \( Rwu \) 이고 \( Rwv \) 인 모든 \( v \) 에 대하여 \( Rwv \) 이고 \( \mathfrak{F} , V , v \models p \) 이다. 따라서 \( \mathfrak{F} , V , w \models \Box \Box p \) 임을 알 수 있다.

"예제 7: \( \langle a \rangle p \to \langle b \rangle p ? \)"

\( \langle a \rangle p \to \langle b \rangle p \) defines \( R_a \subseteq R_b \)
"General frames"

Frames vs. Models
Validity vs. Satisfaction

A general frame \((\mathcal{F}, A)\) is a frame \(\mathcal{F}\) together with a restricted, but suitably well-behaved collection \(A\) of admissible valuations.

"Operations corresponding to modalities"

Given a frame \(= (W, R)\) and \(X \subseteq W\),

\[
m_R(X) = \{ w \in W \mid Rwx \text{ for all } x \in X \}
\]

Note: \(V(\Box \phi) = m_R(V(\phi))\)

Given an \((n + 1)\)-ary relation \(R\) on a set \(W\),

\[
m_R(X_1, \ldots, X_n) = \{ w \in W \mid Rw_1 \ldots w_n \text{ for all } w_1 \in X_1, \ldots, w_n \in X_n \}
\]
"General frames" (formally)

A \textit{general }\tau\text{-frame} \( \mathcal{F} = (W, R_\Delta)_{\Delta \in \tau} \)
and \( A \) is a non-empty collection of \textit{admissible} subsets of \( W \) closed under the following operations:

(i) intersection: if \( X, Y \in A \), then \( X \cap Y \in A \)

(ii) relative complement: if \( X \in A \), then \( W \setminus X \in A \)

(iii) modal operations: if \( X_1, ..., X_n \in A \), then \( m_{R_\Delta}(X_1, ..., X_n) \in A \)
for all \( \Delta \in \tau \)

A \textit{model based on a general frame} is a triple \( (\mathcal{F}, A, V) \) where \( (\mathcal{F}, A) \) is a general frame and \( V \) is a valuation satisfying the constraint that for each proposition letter \( p \), \( V(p) \) is an element of \( A \). Valuations satisfying this constraint are called \textit{admissible} for \( (\mathcal{F}, A) \).

"Local semantic consequence"

\( \phi \) is a \textit{local semantic consequence} of \( \Sigma \) over \( S : \Sigma \models_{S} \phi \)

for all models \( \mathcal{M} \) from \( S \), and all points \( w \) in \( \mathcal{M} \),
if \( \mathcal{M}, w \models \Sigma \) then \( \mathcal{M}, w \models \phi \)

"Global semantic consequence"

\( \phi \) is a \textit{global semantic consequence} of \( \Sigma \) over \( S : \Sigma \models^{g}_{S} \phi \)

for all structures \( \mathcal{G} \) in \( S \),
if \( \mathcal{G} \models \Sigma \) then \( \mathcal{G} \models \phi \)
"System K"

The axioms of K:

- propositional tautologies
- (K) □(p → q) → (□p → □q)

The rules of proof of K:

- Modus ponens: given φ and φ → ψ, prove ψ
  - preserves validity, global truth and satisfaction
- Uniform substitution: given φ, prove θ, where θ is obtained from φ by uniformly replacing proposition letters in φ by arbitrary formulas.
  - preserves only validity
- Necessitation: given φ, prove □φ
  - preserves validity and global truth

"K the minimal axiom system"

A K-proof is a finite sequence of formula, each of which is an axiom, or follows from one or more earlier items in the sequence by applying a rule of proof. (Hilbert style)

A formula φ is K-provable (\vdash_K φ) if it occurs as the last item of some K-proof.

K is sound with respect to the class of all frames
- All K-provable formulas are valid
- its axioms are all valid and all 3 rules of proof preserve validity.

K is complete with respect to the class of all frames
- All valid formulas are K-provable.
"예제 8: \((\Box p \land \Box q) \rightarrow \Box(p \land q)\) is valid?"

1. \(\vdash p \rightarrow (q \rightarrow (p \land q))\)  
   Tautology
2. \(\vdash \Box(p \rightarrow (q \rightarrow (p \land q)))\)  
   Necessitation: 1
3. \(\vdash \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)\)  
   K axiom
4. \(\vdash (\Box(p \rightarrow (q \rightarrow (p \land q)))) \rightarrow (\Box p \rightarrow \Box(q \rightarrow (p \land q)))\)  
   Uniform substitution: 3
5. \(\vdash \Box (p \rightarrow q) \rightarrow (p \land q)\)  
   Modus Ponens: 2,4
6. \(\vdash \Box (q \rightarrow (p \land q)) \rightarrow (\Box q \rightarrow \Box(p \land q))\)  
   Uniform substitution: 3
7. \(\vdash \Box (p \rightarrow \Box q) \rightarrow \Box (p \land q)\)  
   Propositional logic: 5,6
8. \(\vdash (\Box p \land \Box q) \rightarrow \Box(p \land q)\)  
   Propositional logic: 7

Modal logic

"System K4"

The axioms of K4:

- propositional tautologies
- \((K) \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)\)
- \((4) \Box p \rightarrow \Box \Box p\)

The rules of proof of K4:

- Modus ponens
- Uniform substitution
- Necessitation

\[\Sigma \vdash_{K4} \phi \text{ iff } \Sigma \vdash_{\text{Tran}} \phi\]
Modal logic

"Axioms and frame conditions"

<table>
<thead>
<tr>
<th>Name</th>
<th>Axiom</th>
<th>Condition on Frames</th>
<th>( R ) is ...</th>
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<tbody>
<tr>
<td>(D)</td>
<td>( \Box p \rightarrow \Diamond p )</td>
<td>( \exists u. wRu )</td>
<td>Serial</td>
</tr>
<tr>
<td>(M)</td>
<td>( \Box p \rightarrow p )</td>
<td>( wRw )</td>
<td>Reflexive</td>
</tr>
<tr>
<td>(4)</td>
<td>( \Box p \rightarrow \Box \Box p )</td>
<td>if ( (wRv \text{ and } vRu) ) then ( wRu )</td>
<td>Transitive</td>
</tr>
<tr>
<td>(B)</td>
<td>( p \rightarrow \Box \Diamond p )</td>
<td>if ( wRv ) then ( vRw )</td>
<td>Symmetric</td>
</tr>
<tr>
<td>(5)</td>
<td>( \Diamond p \rightarrow \Box \Diamond p )</td>
<td>if ( (wRv \text{ and } wRu) ) then ( vRu )</td>
<td>Euclidean</td>
</tr>
<tr>
<td>(CD)</td>
<td>( \Diamond p \rightarrow \Box p )</td>
<td>if ( (wRv \text{ and } wRu) ) then ( v = u )</td>
<td>Deterministic</td>
</tr>
<tr>
<td>( \Box M )</td>
<td>( \Box (\Box p \rightarrow p) )</td>
<td>if ( wRv ) then ( vRv )</td>
<td>Shift Reflexive</td>
</tr>
<tr>
<td>(C4)</td>
<td>( \Box \Box p \rightarrow \Box p )</td>
<td>if ( wRv ) then ( \exists u. (wRu \text{ and } uRv) )</td>
<td>Dense</td>
</tr>
<tr>
<td>(C)</td>
<td>( \Diamond \Box p \rightarrow \Box \Diamond p )</td>
<td>if ( (wRv \text{ and } wRx) ) then ( \exists u. (vRu \text{ and } xRu) )</td>
<td>Convergent</td>
</tr>
</tbody>
</table>

Modal logic

"Relationships among modal logics"
Modal logic

"Result by Scott and Lemmon"

\[(G)\] \(\Diamond^h \Box^i p \rightarrow \Box^j \Diamond^k p\)

\[(4)\] \(\Box p \rightarrow \Box \Box p = \Diamond^0 \Box^1 p \rightarrow \Box^2 \Diamond^0 p\)

(hijk-Convergence) if \(R^h w v\) and \(R^j w u\) then \(\exists x. (R^i v x \text{ and } R^k u x)\)

(0120-Convergence) if \(R^0 w v\) and \(R^2 w u\) then \(\exists x. (R^1 v x \text{ and } R^0 u x)\)

(transitivity) if \(R v x\) and \(R x u\) then \(R v u\)

Note: Sahlqvist (1975) has discovered important generalizations of the Scott-Lemmon result covering a much wider range of axiom types.

Modal logic

"모달 로직에 관심을 갖게 하는 것들"

- Temporal logics
- Dynamic logics
- Fixpoint logics
- Modal type system
- Model checking
- Semantics
- Bisimulation
- Analysis and Verification of modal properties
- ...

Note: Sahlqvist (1975) has discovered important generalizations of the Scott-Lemmon result covering a much wider range of axiom types.
New typing ideas from (intuitionistic variants of) standard modal logics
Potential applications include type systems for...

- run-time code generation
- meta-programming and higher-order syntax with free-variables
- memoization and incremental computation
- information flow and security
- distributed computation
- resource-bounded computation
- ...

taken from B. Pierce’s slides