Martin-Löf's Type Theory

Bengt Nordström

bengt@cs.chalmers.se

ChungAng University, Seoul, Korea

on leave from Chalmers University, Göteborg, Sweden

The world

Sweden


0 50 100 km
0 50 100 mi
Kiruna
Norwegian Sea
Tärnaby
Luleå
Umeå
Gulf of Bothnia
FINLAND
Sundsvall
Hudiksvall
NORWAY
Gävle
Uppsala
Åland Is. (FINLAND)
Karlstad
Örebro
STOCKHOLM
ESTONIA
Norrköping
Linköping
Jönköping
Göteborg
Gotland
Kattegat
LATVIA
Kalmar
Halmstad
Karlshamn
Öland
Helsingborg
Karlskrona
Baltic Sea
LITH.
Sölvesborg
Malmö
Bornholm (DENMARK)
DENMARK
RUSSIA
GERMANY
POLAND

Background

Martin-Löf developed his type theory during 1970 – 1980 as a foundational language for mathematics. It is based on Constructive Mathematics and a proposition is the set of all its proofs. The following identificatons can be made:

- $a \in A$
- a is a *proof* of the *proposition* A
- a is an *object* in the *type* A
- a is a *program* with *specification* A
- a is a *solution* to the *problem* A

Proofs as Programs

A direct proof of:	consists of:	As a type:
$A \vee B$	a proof of A or a proof of B	**data** Or A B = Ori1 A | Ori2 B;
$A \& B$	a proof of A and a proof of B	**data** And A B = Andi A B;
$A \supset B$	a method taking a proof of A to a proof of B	**data** Implies A B = Impi A -> B;
$Falsity$		**data** Falsity = ;

Constructors are introduction rules

Ori1 $\in \quad A \to A \vee B$ $\qquad \dfrac{A}{A \vee B}$

Ori2 $\in \quad B \to A \vee B$ $\qquad \dfrac{B}{A \vee B}$

Andi $\in \quad A \to B \to A \,\&\, B$ $\qquad \dfrac{A \quad B}{A \,\&\, B}$

Impli $\in \quad (A \to B) \to A \supset B$ $\qquad \dfrac{\begin{matrix}[A] \\ B\end{matrix}}{A \supset B}$

Elimination rules can be defined

orel $\in A \vee B \to (A \to C) \to (B \to C) \to C$

$$\dfrac{A \vee B \quad \begin{matrix}[A] \\ C\end{matrix} \quad \begin{matrix}[B] \\ C\end{matrix}}{C}$$

orel $($**Ori1** $a)\ f\ g = f\ a$

orel $($**Ori2** $b)\ f\ g = g\ b$

Elimination rules can be defined

orel $\in A \vee B \to (A \to C) \to (B \to C) \to C$

$$\frac{A \vee B \quad \begin{matrix}[A]\\ C\end{matrix} \quad \begin{matrix}[B]\\ C\end{matrix}}{C}$$

orel $($**Ori1** $a)\ f\ g = f\ a$

orel $($**Ori2** $b)\ f\ g = g\ b$

$$\frac{A \,\&\, B \quad \begin{matrix}[A, B]\\ C\end{matrix}}{C}$$

andel $\in A \,\&\, B \to (A \to B \to C) \to C$

andel $($**Andi** $a\ b)\ f = f\ a\ b$

Elimination rules can be defined

orel $\in A \vee B \to (A \to C) \to (B \to C) \to C$

$$\frac{A \vee B \quad \begin{matrix}[A]\\ C\end{matrix} \quad \begin{matrix}[B]\\ C\end{matrix}}{C}$$

orel $($**Ori1** $a)\ f\ g = f\ a$

orel $($**Ori2** $b)\ f\ g = g\ b$

$$\frac{A \,\&\, B \quad \begin{matrix}[A, B]\\ C\end{matrix}}{C}$$

andel $\in A \,\&\, B \to (A \to B \to C) \to C$

andel $($**Andi** $a\ b)\ f = f\ a\ b$

$$\frac{A \supset B \quad A}{B}$$

implel $\in A \supset B \to A \to B$

implel $($**Impli** $f)\ a = f\ a$

Proof checking = Type checking

In this way we can prove propositional formulas in a typed functional programming language. The problem of proving for instance

$$(A \mathbin{\&} B) \supset (B \mathbin{\&} A)$$

is then the problem of finding a program in this type. The type checker will check if the proof is correct.
In this case, we can use the following program:

$$\textbf{Impli}\ (\lambda x.\textbf{Andi}\ (\textbf{andel}\ x\ \lambda y.\lambda z.z)\ (\textbf{andel}\ x\ \lambda y.\lambda z.y))$$

Two questions:

- Is is possible to extend this to more powerful logics, like predicate logic?
- We have that a proof of P is an object in the type P.
  But is it possible to identify the *process of proving* P with the *process of building* an object in the type P?

The answer to these two questions is yes.

Overview of type theory

Type theory is a small typed functional language with one basic type and two type forming operation.
It is a **framework** for defining logics.
A new logic is introduced by definitions.

What types are there?

- Set is a type
- If $A \in$ Set then *El*(A) is a type
- If A is a type and B a family of types for $x \in A$ then $(x \in A)B$ is a type.

What programs are there?

Programs are formed from variables and constants using abstraction and application:

- Application

$$\frac{c \in (x \in A)B \quad a \in A}{c\ a \in B[x := a]}$$

- Abstraction

$$\frac{b \in B \quad [x \in A]}{[x]b \in (x \in A)B}$$

- constants are either primitive or defined

Constants

There are two kinds of constants:

primitive: (not defined) have a type but no definiens (RHS):

$$\text{identifier} \in \text{Type}$$

defined: have a type and a definiens:

$$\text{identifier} = \text{expr} \in \text{Type}$$

There are two kinds of defined constants:

- explicitly defined
- implicitly defined

Primitive constants

- computes to themselves (i.e. are values).
- constructors in functional languages.
- introduction rules and formation rules in logic
- postulates

Examples:

$$
\begin{array}{lcl}
\mathsf{N} & \in & \mathsf{Set} \\
0 & \in & \mathsf{N} \\
\mathsf{s} & \in & (\mathsf{N})\mathsf{N} \\
\& & \in & (\mathsf{Set},\ \mathsf{Set})\mathsf{Set} \\
\&\mathsf{I} & \in & (A \in \mathsf{Set},\ B \in \mathsf{Set},\ A,\ B)A \,\&\, B \\
\Pi & \in & (A \in \mathsf{Set},\ (A)\mathsf{Set})\ \mathsf{Set} \\
\lambda & \in & (A \in \mathsf{Set}, B \in (A)\mathsf{Set}, (x \in A)B(x)) \\
 & & \Pi(A, B)
\end{array}
$$

Explicitly defined constants

- have a type and a definiens (RHS).
- the definiens is a welltyped expression
- abbreviation
- derived rule in logic.
- names for proofs and theorems in math.

Examples:

$$
\begin{array}{lcl}
2 & \equiv & \mathsf{succ}(\mathsf{succ}(0)) \in \mathsf{N} \\
\forall & \equiv & \Pi \in (A \in \mathsf{Set},\ (A)\mathsf{Set})\ \mathsf{Set} \\
+ & \equiv & [x, y]\mathsf{natrec}([x]\mathsf{N}, x, y, [u, v]\mathsf{succ}(v)) \in (\mathsf{N}, \mathsf{N})\mathsf{N} \\
\rightarrow & \equiv & [A, B]\Pi(A, [x]B)) \in (A, B \in \mathsf{Set})\mathsf{Set}
\end{array}
$$

Implicitly defined constants

The definiens (RHS) may contain pattern matching and may contain occurrences of the constant itself. The correctness of the definition must in general be decided outside the system

- Recursively defined programs
- Elimination rules (the step from the definiendum to the definiens is the contraction rule).

Examples:

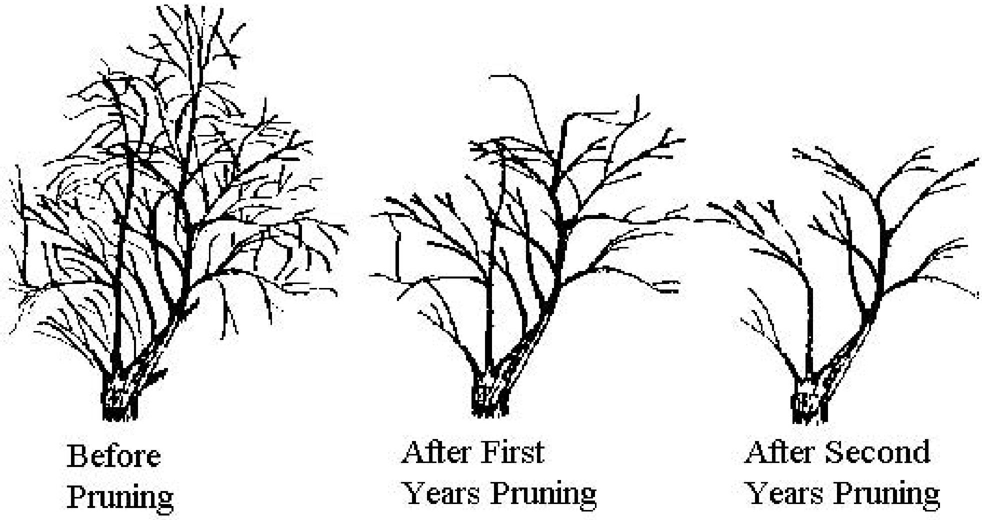
$$\&\mathsf{E} \in (A \in \mathsf{Set},\ B \in \mathsf{Set},\ C \in (A,\ B)\mathsf{Set},$$
$$(x \in A,\ y \in B)C(\&\mathsf{I}(x,y)),\ (z \in A \,\&\, B))C(z)$$
$$\&\mathsf{E}(A, B, C, f, \&\mathsf{I}(a,b)) \equiv f(a,b)$$

The editing process

The idea is to build expressions from incomplete expressions with holes (placeholders). Each editing step replaces a place holder with another incomplete expression

The editing process

The idea is to build expressions from incomplete expressions with holes (placeholders). Each editing step replaces a place holder with another incomplete expression (pruning a tree goes in the other direction).



Place holders

We use the notation

$$\Box_1, \ldots \Box_n$$

for place holders (holes).
Each place holder has an **expected type** and a **local context** (variables which may be used to fill in the hole).

To prove is to build

- To apply a rule **c** is to construct an application of the constant **c**.
- To assume **A** is to construct an abstraction of a variable of type **A**.
- To refer to an assumption of **A** is to use a variable of type **A**.

To construct an object

We start to give the name of the object to define, and the computer responds with

$$c \in \Box_1$$
$$c = \Box_2$$

We must first give the type of c by refining $\Box_1$.
We can either enter text from the keyboard, or do it stepwise, replace it by

- $(x \in \Box_3)\Box_4$ — a function type, or
- Set, or
- $C(\Box_3, \ldots \Box_n)$

Refinement of an object

When we have constructed the type of the constant c, we can start to define it:

$$c \in C$$
$$c = \Box_0$$

Here, *the expected type* of $\Box_0$ is C .
In general, we are in a situation like

$$c = \ldots \Box_1 \ldots \Box_2 \ldots$$

where we know the expected type of the place holders.

Refinement of an object: application

To refine a place holder

$$\Box_0 \in A$$

with a constant c (or a variable) is to replace it by

$$c\ \Box_1\ \ldots\ \Box_n \in A$$

where $\Box_1 \in B_1, \ldots, \Box_n \in B_n$. The system computes n and the expected types of the new place holders as well as some constraints from the condition that the type of $c\ \Box_1\ \ldots\ \Box_n$ must be equal to A.

Refinement of an object: application

To refine a place holder

$$\square_0 \in A$$

with a constant c (or a variable) is to replace it by

$$c\,\square_1\ \ldots\ \square_n \in A$$

where $\square_1 \in B_1, \ldots, \square_n \in B_n$. The system computes n and the expected types of the new place holders as well as some constraints from the condition that the type of $c\,\square_1\ \ldots\ \square_n$ must be equal to A.

We have reduced the problem A to the subproblems B_1, . . . B_n using the rule c.

Refinement of an object: abstraction

To refine a place holder

$$\square_0 \in A$$

with an abstraction is to replace it by

$$[x]\square_1 \in A$$

The system checks that A is a functional type $(x \in B)C$ and the expected type of $\square_1$ is C and the local context for it will contain the assumption $x \in B$.

Refinement of an object: abstraction

To refine a place holder

$$\Box_0 \in A$$

with an abstraction is to replace it by

$$[x]\Box_1 \in A$$

The system checks that A is a functional type $(x \in B)C$ and the expected type of $\Box_1$ is C and the local context for it will contain the assumption $x \in B$.

We have reduced the problem $(x \in B)C$ to the problem C by using the assumption $x \in B$.

Summary

- proof = program (proposition = type): examples in Haskell
- Can this be extended to Predicate Logic?
- Process of proving = process of building a program?

Summary

- proof = program (proposition = type): examples in Haskell
  - Introduction rules are constructors
  - Elimination rules can be defined
  - Proof checking = Type checking
- Can this be extended to Predicate Logic?
- Process of proving = process of building a program?

Summary

- proof = program (proposition = type): examples in Haskell
  - Introduction rules are constructors
  - Elimination rules can be defined
  - Proof checking = Type checking
- Can this be extended to Predicate Logic?
  - Yes, by having types depending on objects.
- Process of proving = process of building a program?

Summary

- proof = program (proposition = type): examples in Haskell
  - Introduction rules are constructors
  - Elimination rules can be defined
  - Proof checking = Type checking
- Can this be extended to Predicate Logic?
  - Yes, by having types depending on objects.
- Process of proving = process of building a program?
  - To apply a rule **c** is to construct an application of the constant **c**.
  - To assume **A** is to construct an abstraction of a variable of type **A**.
  - To refer to an assumption of **A** is to use a variable of type **A**.

Type Theory

- Types:
- Programs:
- Constants:

Type Theory

- Types:

$$T \ ::= \ \textsf{Set} \ | \ \textsf{\textit{El}}(e) \ | \ (x \in T)T'$$

- Programs:
- Constants:

Type Theory

- Types:

$$T \ ::= \ \textsf{Set} \ | \ \textsf{\textit{El}}(e) \ | \ (x \in T)T'$$

- Programs:

$$e \ ::= \ e\, e' \ | \ [x]e \ | \ x \ | \ c$$

- Constants:

Type Theory

- Types:

$$T \ ::= \ \mathsf{Set} \ | \ \mathit{El}(e) \ | \ (x \in T)T'$$

- Programs:

$$e \ ::= \ e\,e' \ | \ [x]e \ | \ x \ | \ c$$

- Constants:
  - Primitive (without a defi nition):

$$c \in T$$

  - Explicitly defi ned:

$$c = e \in T$$

  - Implicitly defi ned:

$$\begin{array}{rcl} c\,p_1 \ \ldots \ p_n & = & e \\ & \vdots & \\ c\,p'_1 \ \ldots \ p'_n & = & e' \end{array}$$